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A Linear Programming Model for Scheduling Crude Oil Production

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INTRODUCTION

In recent years considerable progress has been made in the development and application of mathematical techniques for the solution of certain problems involving economic "strategies". Such a problem might involve, for example, the scheduling of shipments of a commodity from a number of sources to a number of destinations. The object would be to schedule the shipments in a manner so as to satisfy the destination requirements and at the same time minimize the transportation costs. The solution to such a problem is not necessarily intuitively obvious. The "obvious" solution is frequently far from optimum. If the shipments are to be made from, for example, only two sources to four destinations, the optimum schedule is readily found. However, if shipments are to be made from, say, 10 sources to several hundred destinations, even a competent and experienced scheduler may spend considerable time in finding a reasonable answer. Even then he is not sure that he has the optimum solution. Furthermore, he has no way of knowing how far from optimum the answer is. Consequently, he does not know whether he should accept this solution or seek a better one.

Prior to the advent of large high-speed digital computers, little more could be done with such problems because of their great size and multiplicity of possible solutions. A problem involving 20 sources and 50 destinations would require choosing, from a very large number of possible combinations, the optimum combination of 1,000 variables. The best one could do was to utilize intuition, extrapolation from past experience, and other non-exact approaches. With a high-speed computer, however, such problems can be solved providing a reasonable computational procedure (or algorithm) can be utilized. The purpose of this paper is to describe such a procedure (linear programming) and to apply this procedure to a production scheduling problem.

THE RESERVOIR PROBLEM

A simple reservoir model is used throughout this analysis to illustrate the use of linear programming. Even the simplest reservoir behavior problem is nonlinear in both space and time, but if the geometry is fixed for a particular study and the time variable is quantized, the resultant system may be described by linear constraints on the variable production rates.

Crude oil is available from five separate sources and is delivered to a pipeline. Sources 1, 2, 3 and 4 are ideal reservoirs and therefore are subjected to reservoir flow restrictions. Source 5 is labeled "Outside Source" and crude oil from this source is assumed to be available in unlimited quantities without considering reservoir conditions. The general problem is to determine the schedule of crude oil production from five sources which, over an eight-year period and subject to certain restrictions, will result in maximum profit.

"Profit" is defined to encompass all economic factors and expenses involved in producing and selling crude oil to a pipeline facility. Such economic factors are lumped to the extent that the term represents the dollars profit from one barrel of crude oil from any of the five sources entering the pipeline system. Table 1 presents the assumed profit per barrel for oil from any of the five sources over each of the four time periods. In this chart the eight years is broken into four equal intervals of two years. The selection of four time periods of two years each and the values representing the potential profit are completely arbitrary selections for the purpose of illustrating this idealized reservoir problem. In any practical application of this work a careful economic study would have to be made in order to estimate potential unit profits for time periods as far in the future as eight years. In this particular study it is assumed that all of the potential unit profits (C_{ij}) are known and fixed before the production schedule is attempted.

For each reservoir arbitrary values are assigned for

TABLE 1—POTENTIAL PROFIT CHART—SINGLE WELL SYSTEM
(C_{ij} = potential profit, $/bbl)

Reservoir					Outside source
Time	1	2	3	4	5
1	0.10	0.13	0.20	0.40	0.05
2	0.17	0.18	0.23	0.46	0.09
3	0.20	0.22	0.30	0.49	0.16
4	0.23	0.28	0.36	0.52	0.20

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permeability, viscosity, outer radius, initial pressure, etc. It is assumed that all of the reservoir parameters remain constant throughout the eight-year period of study. It is further assumed that each reservoir contains a single ideal liquid in a homogeneous porous medium under the influence of an infinite water drive. An additional assumption is that the radial flow formulas for infinite water drive are applicable.[1,2] For such an infinite water-drive system, the following equation represents the relation between well pressure and production rate.

$$P_w(\tau) = P_o - \frac{\mu}{2\pi hk} QP(\tau) \; , \quad . \quad . \quad . \quad . \quad (1)$$

where τ represents dimensionless time, P_w is well pressure, P_o the initial pressure of the reservoir, and $P(\tau)$ is the function developed and described by Van Everdingen and Hurst[2]. Time is a continuous variable and Eq. 1 determines the well pressure at any instant of time provided the production rate, Q, the initial pressure, P_o, and the other physical parameters are known for this reservoir.

Now the pressure in the j^{th} reservoir at the end of the i^{th} time period may be written,

$$P_{wj}(\tau_{ij}) = P_{oj} - \frac{\mu_j}{2\pi h_j k_j} \sum_{k=1}^{i}$$

$$(Q_{kj} - Q_{k-1,j}) P(\tau_{ij} - \tau_{k-1,j}) \; , \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where Q_{ij} is the average production rate of the j^{th} reservoir during the i^{th} time period, P_{oj} is initial pressure of the j^{th} reservoir, τ_{ij} is dimensionless time and $P(\tau_{ij}) = \frac{1}{2}(\ln \tau_i + .80907)$, $\tau_i \geq 1{,}000$.

The well pressures (in all four reservoirs) are not allowed to go below some arbitrary value, such as 1 atm. Thus, there are 16 linear constraints on the production rates, Q_{ij} (one for each reservoir for each of the four time periods). The set of 16 inequalities is represented by

$$P_{wj}(\tau_{ij}) \geq 1 \qquad i,j = 1(1)\,4. \quad . \quad . \quad . \quad . \quad (3)$$

The next set of constraints considered is represented by

$$\sum_{i=1}^{4} Q_{ij}\,\Delta t \leq F_j \qquad j = 1(1)\,4 \; , \quad . \quad . \quad . \quad . \quad (4)$$

where F_j represents the original volume of oil in place in the j^{th} reservoir.

An additional set of four constraints is represented by Ineq. 5, which states that the production rate in any time period may not exceed the pipeline capacity, R.

$$\sum_{j=1}^{5} Q_{ij}\,\Delta t \leq R \qquad i = 1(1)\,4 \; , \quad . \quad . \quad . \quad (5)$$

where Q_{ij} is the volume rate of oil purchased from the outside source in the i^{th} time period.

The set Ineq. 3 contributes 16 inequalities which constrain the individual reservoir production rates. Ineq. 4 contributes four inequalities which constrain the cumulative production from each reservoir. Ineq. 5 contributes four inequalities which represent the pipeline limitations. In total there are 24 linear inequalities expressed in terms of the unknown, Q_{ij}. The problem then is to find values of Q_{ij} that will satisfy Ineqs. 3, 4 and 5 with the additional requirement that the chosen Q_{ij} shall result in maximum profit. That is,

$$\sum_{j=1}^{5} \sum_{i=1}^{4} Q_{ij}\,\Delta t\, C_{ij} = \text{Maximum} \; . \quad . \quad . \quad (6)$$

C_{ij} represents the potential unit profit in the i^{th} time period for the oil produced from the j^{th} reservoir as given in Table 1.

The problem has been described, therefore, in terms of the maximization of a linear functional whose variables are subject to linear constraints.

RESULTS FOR SINGLE WELL SYSTEM

Tables 2 through 5 show the results of four different studies of this model. Table 2 presents the optimum schedule of production rates for each time period and for all five sources. The profit of $27.6 × 10[6] is the maximum profit that can be realized for this case. Also listed in Table 2 is the cumulative oil produced, the initial oil in place, and the recovery from each of the

TABLE 2—CASE 1
(Production Rates, cm³/sec)

	Source				
Time	1	2	3	4	5
1					102,606
2	1,008	4,834		51,303	45,461
3					102,606
4			11,414		91,192
Cumulative production (cm³) × 63.12 × 10⁶	1,008	4,834	11,414	51,303	341,865
Oil in place (cm³) × 63.12 × 10⁶	1,008	4,834	11,414	51,303	
Per cent recovery	100	100	100	100	

Profit = $27.6 × 10⁶
R = 102,606 cm³/sec
Each time period is two years

TABLE 3—CASE 2
(Production Rates cm³/sec)

	Source				
Time	1	2	3	4	5
1	1,015	63	11,403	51,062	39,063
2	985	4,842	156	49,707	46,917
3	16	0	0	1,837	100,752
4	0	4,763	11,269	0	86,574
Cumulative production (cm³) × 63.12 × 10⁶	2,016	9,668	22,828	102,606	273,306
Oil in place (cm³) × 63.12 × 10⁶	2,016	9,668	22,828	102,606	
Per cent recovery	100	100	100	100	

Profit = $35.1 × 10⁶
R = 102,606 cm³/sec

TABLE 4—CASE 3
(Production Rates cm³/sec)

	Source				
Time	1	2	3	4	5
1	1,015	4,844	11,403	51,062	34,282
2	985	4,707	11,087	49,707	36,120
3	968	4,631	10,911	48,950	37,146
4	957	4,578	10,790	48,429	37,852
Cumulative production (cm³) × 63.12 × 10⁶	3,925	18,760	44,191	198,148	145,400
Oil in place (cm³) × 63.12 × 10⁶	4,032	19,336	45,656	205,212	
Per cent recovery	97	97	97	97	

Profit = $41.3 × 10⁶
R = 102,606 cm³/sec

TABLE 5—CASE 4
(Production Rates cm³/sec)

	Source				
Time	1	2	3	4	5
1	1,015	4,844	11,403	51,062	34,282
2	985	4,707	11,087	49,707	36,120
3	968	4,631	10,911	48,950	37,146
4	957	4,578	10,790	48,429	37,852
Cumulative production (cm³) × 63.12 × 10⁶	3,925	18,760	44,191	198,148	145,400
Oil in place (cm³) × 63.12 × 10⁶	4,032	38,672	136,968	205,212	
Per cent recovery	97	49	32	97	

Profit = $41.3 × 10⁶
R = 102,606 cm³/sec

[1]References given at end of paper.

four reservoirs. Recovery here is defined to be the fraction of the initial oil in place that has been produced. In the first case the pressure constraints represented by Ineq. 3 played no role because it was not possible to withdraw at such a high rate that the well pressure would be reduced to 1 atm. Since the constraints of Ineq. 3 play no role in Case 1, these pressure constraints could be completely eliminated from this particular case. As there are only eight remaining constraints, there can be only eight non-zero values of the 20 Q_{ij}.

In Case 2, the F_j's are doubled. In certain instances some of the constraints of Ineq. 3 are tight, i.e., equalities. For example, in Reservoir 1 the well pressure at the end of the first time period has dropped to 1 atm.

In Case 3 we again double the F_j's. The recovery is 97 per cent from all four reservoirs; however, production is scheduled for all four time periods from the five sources. In addition, the well pressure has been reduced to 1 atm in every reservoir at the end of all four time periods. The pressure distribution for Reservoir 1, Case 3, is plotted in Fig. 1. The well pressure drops from the initial reservoir pressure to 1 atm at the conclusion of the first time period. The well pressure then rises as the rate is reduced and falls to 1 atm at the conclusion of the second time period. This pattern is repeated for the third and fourth time periods.

In the next case (Case 4), F_2 is again doubled, and F_3 is tripled. The results for this case are given in Table 5. For Reservoirs 2 and 3, the recovery is now 49 and 32 per cent, respectively. This analysis is valid for a single well in each reservoir. The next section presents a method for treating multi-well systems.

MULTI-WELL SYSTEM

Prior to designing a linear programming model for a multi-well system it is necessary to examine the set of constraints in Ineq. 3. These constraints were originally developed by considering the relation between well pressure and production rates for a single well system. In a multi-well system with interference, it is necessary to have equations that relate the well pressure to production rates when more than one well is being produced. Reservoir 3 now has two wells, designated as Well a and Well b, separated by distance, d. r_{ab} represents the distance, d, divided by the well radius. It is assumed that the compressibility and viscosity of the water are the same as that of the oil, so it is permissible to use a superposition principle to relate well pressures and rates (infinite system). Such an analysis has been described in detail and solutions are available.[3]

The pressure in Well a at the end of the i^{th} time period is

$$P_{w.3a}(\tau_i) = \left[P_{o.3a} - A_3 \left\{\sum_{k=1}^{i}\left(Q_{k.3a}[P(1,\tau_i - \tau_{k-1}) - P(1, \tau_i - \tau_k)] + Q_{k.3b}[P(r_{ab}, \tau_i - \tau_{k-1}) - P(r_{ab}, \tau_i - \tau_k)]\right)\right\}\right] \geq 1, \; i = 1(1)\;4 , \quad . \quad . \quad . \quad . \quad . \quad (7)$$

where P_{03} is initial pressure, A_3 is $\mu_3/2\pi h_3 k_3$, Q_{i3a} is production rate during i^{th} time period from Well 3_a, Q_{i3b} is production rate during i^{th} time period from Well 3_b, τ is dimensionless time, and r_{ab} is well spacing ratio $= \dfrac{\text{distance between the two well centers}}{\text{well radius}}$.

$$P(r_{ab},\tau) = \tfrac{1}{2}\left[\ln \frac{4\tau}{r_{ab}^2} - 0.57722\right], \; \frac{4\tau}{r_{ab}^2} \geq 2{,}000 \; . \quad . \quad . \quad . \quad . \quad (8)$$

There are multi-well reservoir systems where it would not be possible to express the function, $P(r_{ab},\tau)$, as an analytical expression. In such cases one can approximate the P function either by numerical computations on a digital computer or analog solutions on an electric analyzer.

Similar constraints may be written for Well 3_b. The constraints on Reservoirs 1, 2 and 4 will remain as before. The potential unit profits are assumed to be the same for both 3_a and 3_b, and equal to the value assumed for Reservoir 3 in the one-well system. The potential unit profits for Sources 1, 2, 4 and 5 remain the same as in the single well case. It would be a simple matter to modify the potential unit profit values for Reservoir 3 to take into consideration the additional cost required to drill more wells, but such factors have not been considered in this study.

The results for Case 5 are given in Table 6. The well spacing ratio, r_{ab}, and the F_j's and R are the same as those for Case 4.

In Case 5 the additional well increased the cumulative production for Reservoir 3 by 9 per cent to total cumulative recovery of 41 per cent.

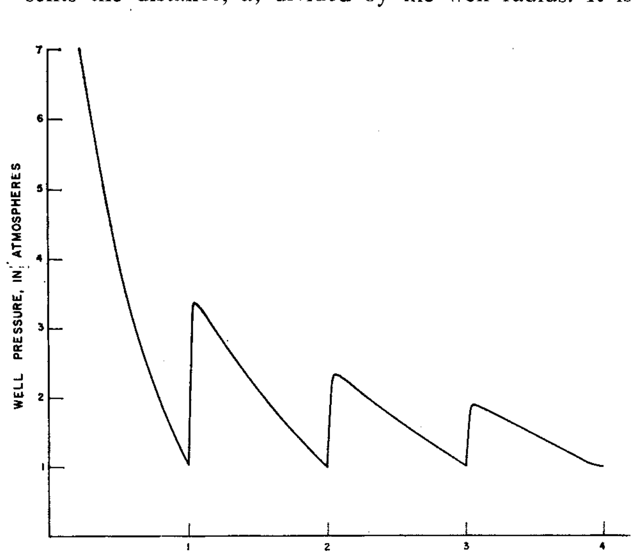

FIG. 1—RESERVOIR 1, CASE 3.

CONCLUSIONS

It is worthwhile to examine the limitations of the present model. One very obvious shortcoming is that only four time periods were considered. It would be much more realistic if there were 20 or more time periods rather than four. Then the well pressure could

TABLE 6—CASE 5
(Production Rates cm³/sec)

Time	Source 1	2	3a	3b	4	5
1	1,015	4,844	7,316	7,316	51,062	31,053
2	985	4,707	7,056	7,056	49,707	33,095
3	968	4,631	6,913	6,913	48,950	34,231
4	957	4,578	6,815	6,815	48,429	35,012
Cumulative production (cm^3) $\times 63.12 \times 10^6$	3,925	18,760	56,200		198,148	133,391
Oil in place (cm^3) $\times 63.12 \times 10^6$	4,032	38,672	136,968		205,212	
Per cent recovery	97	49	41		97	

Profit = $\$49.0 \times 10^6$
R = 102,606 cm^3/sec
r_{ab} = 250

decrease to 1 atm at 20 points instead of only four points. In Fig. 1, the well pressure was reduced to 1 atm at only four points, whereas in a real reservoir system the well pressure could remain at 1 atm for the entire time period of eight years. Use of a larger number of time periods would not make the analysis more difficult, but it would increase the computing time. If, for instance, we consider 20 time periods, then we would have a system of 124 linear constraints on 120 variables. Another limitation is the fact that we considered only a single well and then a two-well system. However, methods are indicated for the introduction of two, three, or any number of wells following this procedure. The inclusion of additional wells merely increases the size of the linear programming model.

No attempt is made in this paper to evaluate all of the important reservoir considerations. In most cases it would be extremely difficult to present a linear relation between well pressure and rates. Even in the case where linear relations can be deduced it is doubtful whether analytical solutions would be available relating well pressure, rate, and time. However, such relations between well pressure, rate and time can be approximated either by analog devices or independent and separate computations on digital computers. Economic factors, which would play an important role in any practical situation, are not emphasized in this study.

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