Unsteady-State Liquid Flow Through Porous Media Having Elliptic Boundaries

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INTRODUCTION

A large number of boundary value problems encountered in unsteady-state heat transfer, fluid flow through porous media, neutron diffusion and mass transfer involve the solution of a linear, parabolic partial differential equation commonly referred to as the diffusivity equation,

$$\nabla^2 U = \frac{1}{K}\frac{\partial U}{\partial t} \quad \ldots\ldots\ldots\ldots \quad (1)$$

where U is the dependent potential variable, K is the diffusivity (hydraulic, thermal, neutron, etc.) and t is the time variable. Solutions to Eq. 1 are available in the literature for a wide variety of initial and boundary conditions.[1-4] The great majority of these solutions are obtained for geometric boundaries corresponding to linear, cylindrical or spherical flow models.

A typical engineering application where the solution to Eq. 1 is required is the calculation of underground water encroachment across the boundaries of oil or natural gas reservoirs. In this particular area of application, the reservoir boundary is invariably approximated by circular geometry. However, the areal shape of many reservoirs can be better approximated by elliptic rather than circular boundaries. Thus the need for a general method of solving the diffusivity equation in elliptic coordinates arises in this problem as well as in other engineering applications involving elliptic boundaries.

The solution to the diffusivity equation usually involves the Error Function for the linear flow model, Bessel functions for the radial flow model and trigonometric or Legendre functions for the spherical flow model. It is well known that the general solution to the diffusivity equation in elliptic coordinates involves Mathieu functions. The significance of Mathieu functions in the analytical treatment of the diffusivity equation in elliptic coordinates is discussed in the literature.[5-7] However, these references do not provide analytical solutions useful in practical engineering problems.

The objectives of this paper are the development of the equations describing the unsteady-state liquid flow through a porous medium with an elliptic inner boundary, the development of a numerical method of solving these equations and, finally, a comparison of the water encroachment quantities calculated from the elliptic flow equation with those calculated from the radial flow equation. While the specific problem treated in this paper relates to unsteady-state liquid flow through a porous medium, the basic equations and computational techniques developed will apply equally well to problems occurring in the other areas of engineering interest mentioned previously. The solution given here is limited to a single case in which the outer boundary encloses an area 100 times that of the inner boundary.

DESCRIPTION OF THE FLOW MODEL

Fig. 1 shows the flow model upon which the calculations presented in this paper are based. The inner and outer boundaries of the flow model are represented by two confocal ellipses with major and minor axes, respectively, equal to $2a_i$, $2b_i$, and $2a_e$ and $2b_e$. The height of the elliptic cylinder flow model is denoted by h.

The following assumptions are employed in the development of the equations governing the unsteady-state liquid flow through the described flow model.

1. Uniform porosity and permeability throughout the flow model

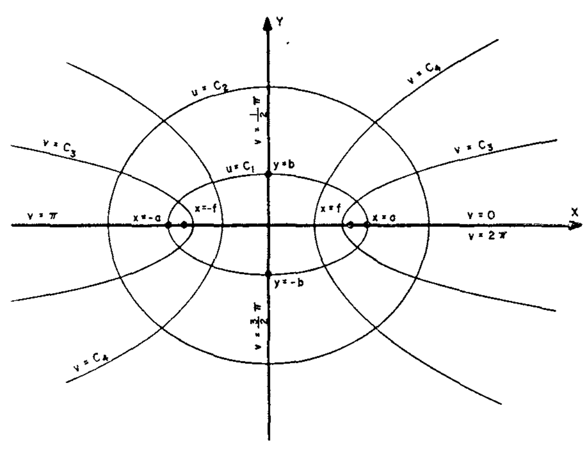

FIG. 1—GEOMETRIC RELATIONSHIPS BETWEEN CARTESIAN AND ELLIPTIC PLANAR COORDINATES.

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[1]References given at end of paper.

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2. Isothermal flow

3. Two-dimensional flow in the horizontal plane, i.e., no flow in the vertical direction.

THE DIFFUSIVITY EQUATION IN ELLIPTIC COORDINATES

The diffusivity equation governing unsteady-state liquid flow through a porous medium has been derived in the literature[3] and is given here as Eq. 2.

$$\nabla^2 p = \frac{\mu\phi c}{k}\frac{\partial p}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where p = liquid pressure, psia,

c = sum of liquid and porous medium compressibilities, vol/vol − psia,

$\frac{k}{\mu}$ = porous medium mobility, ft²/sec-psia,

ϕ = porous medium porosity, fraction.

The form of the term $\nabla^2 p$ is determined by the geometry of the particular flow model being considered. For example if the flow model is a circular cylinder and if the flow is assumed radial, then Eq. 2 becomes

$$\nabla^2 p = \frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{\mu\phi c}{k}\frac{\partial p}{\partial t} \quad . \quad . \quad . \quad . \quad (3)$$

where r is the radius from the center of the cylindrical flow model.

The form of the diffusivity equation governing unsteady-state liquid flow in a porous medium having elliptic boundaries is obtained by expressing $\nabla^2 p$ in elliptic coordinates. The general expression for the three-dimensional Laplacian of a dependent variable $p(u, v, w)$ in curvilinear coordinates u, v, and w is[3,7]

$$\nabla^2 p(u, v, w) = \frac{1}{\alpha\beta\gamma}\left[\frac{\partial}{\partial u}\left(\frac{\beta\gamma}{\alpha}\frac{\partial p}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{\gamma\alpha}{\beta}\frac{\partial p}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{\alpha\beta}{\gamma}\frac{\partial p}{\partial w}\right)\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where

$$\alpha = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2} \quad , \quad . \quad . \quad (5)$$

$$\beta = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2} \quad , \quad . \quad . \quad (6)$$

$$\text{and } \gamma = \sqrt{\left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2} \quad . \quad . \quad (7)$$

The following relations between elliptic and cartesian coordinates can be used to determine α, β and γ from Eqs. 5, 6 and 7

$$x = f\cosh(u)\cos(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

$$y = f\sinh(u)\sin(v) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$z = w \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)$$

Substitution of α, β and γ into Eqs. 4 and 2 then yields

$$\frac{\partial^2 p}{\partial u^2} + \frac{\partial^2 p}{\partial v^2} = [\cosh^2(u) - \cos^2(v)]f^2\frac{\mu\phi c}{k}\frac{\partial p}{\partial t} \quad . \quad (11)$$

Eq. 11 is the diffusivity equation relating the dependent variable p to the elliptic planar coordinates u and v and the time variable t. The term $\frac{\partial^2 p}{\partial w^2}$ does not appear in equation because of Assumption 3, previously stated.

The geometric relationships between the cartesian planar coordinates x and y and the elliptic planar coordinates u and v are shown in Fig. 1. The confocal ellipses (along which u is constant) and the confocal hyperbolas (along which v is constant) are mutually orthogonal or perpendicular to one another at points of intersection just as the lines x = constant and y = constant are mutually orthogonal in the cartesian coordinate system.

EQUATION DESCRIBING THE LIQUID FLOW ACROSS AN ELLIPTIC BOUNDARY

Darcy's flow equation relates the superficial fluid velocity in a porous medium to the pressure gradient in the manner

$$\vec{V} = -\frac{k}{\mu}\vec{\nabla}\left(p + \frac{\rho'}{144}w\right) \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where $\vec{V}$ is the velocity vector, ft/sec, ρ' is the liquid specific weight, lb force/cu ft and w is the vertical distance coordinate, feet. The gradient of the dependent variable $p(u, v, w)$ is expressed in the elliptic coordinates u, v and w as

$$\vec{\nabla} p = \frac{1}{\alpha}\frac{\partial p}{\partial u}\vec{i} + \frac{1}{\beta}\frac{\partial p}{\partial v}\vec{j} + \frac{1}{\gamma}\frac{\partial p}{\partial w}\vec{k} \quad . \quad (13)$$

Substitution of the previously determined expressions for α, β and γ yields

$$\vec{\nabla} p = \frac{1}{f\sqrt{\cosh^2 u - \cos^2 v}}\left(\frac{\partial p}{\partial u}\vec{i} + \frac{\partial p}{\partial v}\vec{j}\right) + \frac{\partial p}{\partial w}\vec{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Substituting $\vec{\nabla} p$ from Eq. 14 into Eq. 12, one obtains

$$\vec{V} = -\frac{k}{\mu}\left[\frac{1}{f\sqrt{\cosh^2 u - \cos^2 v}}\left(\frac{\partial p}{\partial u}\vec{i} + \frac{\partial p}{\partial v}\vec{j}\right) + \frac{\partial p}{\partial w}\vec{k} + \frac{\rho'}{144}\vec{k}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

At any given point on an ellipse, u = constant, the term $\frac{\partial p}{\partial v}\vec{j}$ is proportional to the velocity vector component in the v direction, or in the direction tangent to the ellipse at that point. The term $\frac{\partial p}{\partial v}\vec{j}$ can therefore be deleted from Eq. 15 in this case, since the flow *across* elliptic boundary u = constant is being considered. Also, the term $\frac{\partial p}{\partial w}\vec{k} + \frac{\rho'}{144}\vec{k}$ in Eq. 15 can be set equal to zero since $\frac{\partial p}{\partial w} = -\frac{\rho'}{144}$ from Assumption 3. Making these simplifications in Eq. 15, one obtains

$$V(v, t) = \frac{k}{\mu}\left[\frac{1}{f\sqrt{\cosh^2 u - \cos^2 v}}\frac{\partial p}{\partial u}\right]_{u=K} . \quad (16)$$

where K = a constant and

$V = V(v,t)$ = fluid velocity in negative u direction across the elliptic boundary $u = K$, ft/sec.

The volumetric rate of liquid flow across an infinitesimal area element, dA, at the elliptic boundary is simply VdA, or

$$dq = \frac{k}{\mu}\left[\frac{dA}{f\sqrt{\cosh^2 u - \cos^2 v}}\frac{\partial p}{\partial u}\right]_{u=K} . \quad . \quad . \quad . \quad (17)$$

where $dA = hds$, sq ft,

s = arc length on ellipse $u = K$, ft,

$q = q(t)$ = volumetric liquid flow rate across entire elliptic boundary, $u = K$, cu ft/sec

The differential arc length ds is given on the ellipse by

$$ds = \beta dv = f\sqrt{\cosh^2 u - \cos^2 v}\ dv \quad . \quad . \quad . \quad (18)$$

and substitution of $hf\ \sqrt{\cosh^2 u - \cos^2 v}$ for dA in Eq. 17 yields

$$dq = \frac{kh}{\mu}\left(\frac{\partial p}{\partial u}\right)_{u=K} \mathrm{dv} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

Because of flow symmetry about the x and y axes (see Fig. 1), q can be obtained by integrating dq over the first quandrant from $v = 0$ to $v = \pi/2$ and multiplying the result by four.

$$q(t) = \frac{4kh}{\mu}\int_{v=0}^{v=\pi/2}\left(\frac{\partial p}{\partial u}\right)_{u=K} dv \quad . \quad . \quad . \quad . \quad (20)$$

A dimensionless water influx term, $\overline{Q}_{t_{DE}}$, can now be defined as

$$\overline{Q}_{t_{DE}} = \int_0^{t_{DE}} \overline{q}(t_{DE})\,dt_{DE} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

where $t_{DE} = \dfrac{kt}{\mu\phi cf^2}$ = dimensionless time for elliptic flow

and $\overline{q}(t_{DE}) = \displaystyle\int_{v=0}^{v=\pi/2}\left(\frac{\partial p}{\partial u}\right)_{u=K} dv$. The actual cubic feet of cumulative water flow across the elliptic boundary up to time t is related to $\overline{Q}_{t_{DE}}$ as

$$Q_{tE} = 4h\phi cf^2\Delta p\overline{Q}_{t_{DE}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

NUMERICAL SOLUTION OF THE DEVELOPED EQUATIONS

Definition of the new variables,

$$t_{DE} = \frac{kt}{\mu\phi cf^2}$$

and

$$\overline{p} = p_o - p,$$

simplifies the diffusivity Eq. 11 to Eq. 23.

$$\frac{\partial^2\overline{p}}{\partial u^2} + \frac{\partial^2\overline{p}}{\partial v^2} = \left[\cosh^2(u) - \cos^2(v)\right]\frac{\partial\overline{p}}{\partial t_{DE}} \quad . \quad (23)$$

This equation has been solved numerically for initial and boundary conditions specifying an initial uniform pressure drop of zero throughout the flow model, a pressure drop of one for all time at the gas bubble boundary $u = u_b$, no flow across the aquifer exterior boundary $u = u_e$, and no flow (because of symmetry) across the portion of the x axis ($a_i \leqslant x \leqslant a_e$) represented by $v = 0$ and across the portion of the y axis ($b_i \leqslant y \leqslant b_e$) represented by $v = \pi/2$ (Fig. 1).

The alternating-direction implicit difference method, proposed by Peaceman and Rachford,[8] has been employed in solving Eq. 23 for the above initial and boundary conditions. In applying this method one obtains at each time step a system of simultaneous difference equations. A technique given by Richtmyer[9] has been employed to solve this system of equations on an IBM 704 digital computer.

The term $\overline{Q}_{t_{DE}}$ has been calculated as a function of t_{DE} for a selected elliptic cylinder flow model. The inner boundary of this model was specified as the ellipse $u = u_b = 0.4$. This u value corresponds to an eccentricity of 0.925 or a ratio of 2.63 between major and minor axes. The exterior closed boundary was taken as the confocal ellipse on which $u = u_e = 2.6$. The calculated $\overline{Q}_{t_{DE}}$ values are listed in Table 1 and are plotted in Fig. 2 as a function of dimensionless time t_{DE}. The calculations were programed in the FORTRAN compiler code and were carried out by an IBM 704 digital computer.*

The area ratio of the elliptic flow model considered here is 101.3, where the area ratio is defined as the area included within the exterior boundary divided by the area of an ellipse is πab where the semi-major axis (a) is $f(\cosh u)$ and the semi-minor axis (b) is $f(\sinh u)$, the 101.3 value is obtained as

$$\frac{\pi a_e b_e}{\pi a_i b_i} = \frac{\cosh(2.6)\sinh(2.6)}{\cosh(.4)\sinh(.4)} = 101.3\,.$$

COMPARISON BETWEEN ELLIPTIC AND RADIAL FLOW

Van Everdingen and Hurst[2] have treated the case of unsteady-state liquid flow in a porous circular cylinder model. They solved the diffusivity equation governing radial flow and presented tables of a dimensionless production quantity, $\overline{Q}_{t_D}$. Fig. 2 shows $\overline{Q}_{t_D}$ plotted vs t_D where $t_D = \dfrac{kt}{\mu\phi cr_b^2}$ = dimensionless time for radial flow, for the case of a flow model having an exterior radius 10 times the interior radius. The initial and boundary conditions employed in calculating these particular $\overline{Q}_{t_D}$ values are identical to the conditions used here in solving the elliptic flow diffusivity equation.

The basis of comparison between elliptic and radial flow cannot be equal distances between the interior and exterior boundaries of the elliptic and radial flow models because the former model has no single dimension analogous to the radius of the latter. Comparison has therefore been made on a basis of equal areas encompassed by the exterior ellipse and the exterior circle and

*This program can be obtained as a printed listing of instructions by writing to the authors.

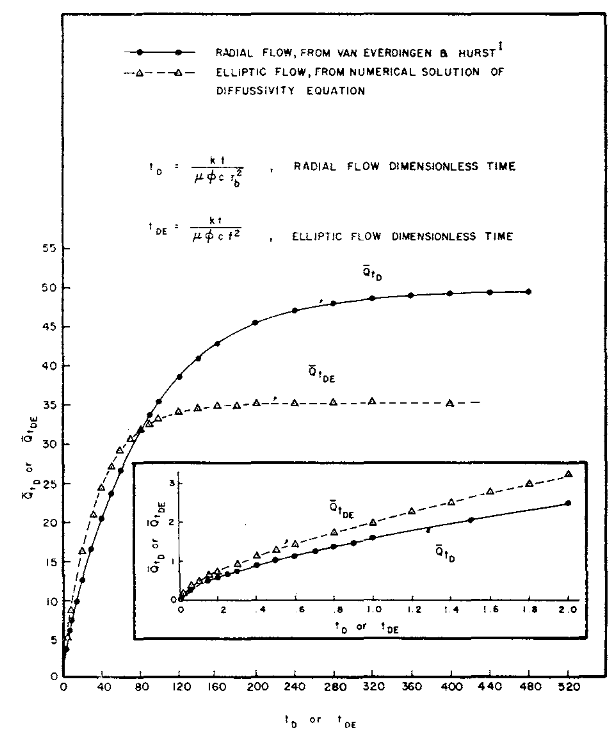

FIG. 2—$\overline{Q}_{t_D}$ AND $\overline{Q}_{t_{DE}}$ VS DIMENSIONLESS TIME.

equal areas included within the interior ellipse and the interior circle. Thus, for equal thicknesses, each flow model contains the same volume (or mass) of water. The equality between the areas included within the interior boundaries yields the relationship

$$\pi r_b^2 = \pi a_i b_i = \pi (f \cosh u_b)(f \sinh u_b) = .445 f^2 \quad . \quad (24)$$

where the area of the interior ellipse is $\pi a_i b_i$ or $0.445 f^2$ since $u_b = .4$ for the case considered here. Eq. 24 is employed in relating t_D to t_{DE} as

$$t_D = \frac{kt}{\mu \phi c r_b^2} = 2.245 \frac{kt}{\mu \phi c f^2} = 2.245 t_{DE} \quad . \quad . \quad (25)$$

Corresponding $\overline{Q}_{t_D}$ and $\overline{Q}_{t_{DE}}$ values should therefore be taken at $t_D = 2.245 t_{DE}$ rather than at $t_D = t_{DE}$.

Equality of the areas enclosed by the exterior boundaries is assured by equal area ratios for the elliptic and radial flow models, provided equal areas are included within the corresponding interior boundaries. The area ratio of the elliptic flow model considered here is 101.3, as mentioned above. However, radial flow $\overline{Q}_{t_D}$ quantities are not tabulated in the literature for this ratio, and available $\overline{Q}_{t_D}$ values corresponding to a ratio of 100 (exterior radius equal to 10 times interior radius) have therefore been used here.

The actual cubic feet of cumulative water influx into the circular sink, Q_t, is related to $\overline{Q}_{t_D}$ as

$$Q_t = 2\pi h \phi c r_b^2 \Delta p \overline{Q}_{t_D} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26)$$

Thus, a comparison between the actual water influx into the elliptic sink and that calculated by approximating the ellipse as an equal area circle and employing the radial flow equation is afforded by the ratio

$$\frac{Q_t}{Q_{t_E}} = \frac{2\pi h \phi c r_b^2 \Delta p \overline{Q}_{t_D}}{4 h \phi c f^2 \Delta p \overline{Q}_{t_{DE}}} = 0.699 \frac{\overline{Q}_{t_D}}{\overline{Q}_{t_{DE}}} \quad . \quad . \quad . \quad (27)$$

where Q_{t_E} is given as $4 h \phi c f^2 \Delta p \overline{Q}_{t_{DE}}$ by Eq. 22. The equality, $Q_t / Q_{t_E} = 1$, would denote exact duplication of the elliptic flow results by the radial flow results for an equal area circle.

The ratio, Q_t / Q_{t_E}, has been calculated as a function of dimensionless time, t_{DE}, and is tabulated in Table 1 and plotted in Fig. 3. Errors in the computed $\overline{Q}_{t_{DE}}$ values and in the $\overline{Q}_{t_D}$ quantities contribute to the tabulated and plotted error in Q_t / Q_{t_E}. Fig. 3 shows that application of radial flow calculations to the elliptic flow case results in an error of the order of 7 per cent in Q_t for small dimensionless time. The error decreases as dimensionless time increases and approaches zero (i.e., Q_t / Q_{t_E} approaches 1.0) for large time. This approach of Q_t / Q_{t_E} to one for large time is a good check on the accuracy of the calculated $Q_{t_{DE}}$ values, since for large time the pressure drop approaches a steady-state uniform value of one, and the total expansion of the equal volumes of water in both flow models (i.e., the cumulative influxes Q_t and Q_{t_E}) should be identical.

TABLE 1—$\overline{Q}_{t_{DE}}$ AND $Q_{t_{DE}}$ AS FUNCTIONS OF DIMENSIONLESS TIME, t_{DE}

$\frac{Q_t}{Q_{t_E}} = 0.699 \frac{\overline{Q}_{t_D}}{\overline{Q}_{t_{DE}}}$, $t_D = 2.245\ t_{DE}$, Er_1 = max. percentage error in $\overline{Q}_{t_{DE}}$, Er_2 = max. percentage error in Q_t/Q_{t_E}

						Er₂, Per cent	
t_{DE}	$\overline{Q}_{t_{DE}}$	Er₁ Per cent	t_D	$\overline{Q}_{t_D}$*	Q_t/Q_{t_E}	Max. Pos. Error	Max. Neg. Error
0	0	0					
.1	.4758	4.4					
.5	1.2920	1.9					
1	1.9892	.98	2.245	2.636	.928	.172	0.98
2	3.1614						
3	4.1801	.50	6.75	5.598	.936	.172	.50
4	5.1163	.59					
6	6.8388	.42	13.5	9.213	.944	.172	.42
8	8.4279						
10	9.9166	.35	22.45	13.42	.948	.192	.35
15	13.2753						
20	16.1852	.30	45	22.09	.955	.288	.30
25	18.7080						
30	20.8955						
35	22.7921						
40	24.4366	.29					
45	25.8625						
50	27.0988	.29	1121.1	37.31	.965	.377	.29
55	28.1707						
60	29.1001						
65	29.9059						
70	30.6046						
75	31.2104						
80	31.7357	.30	180	44.21	.975	.488	.30
85	32.1911						
90	32.5860						
95	32.9284						
100	33.2252						
110	33.7057		247	47.12	.980	.713	.31
120	34.0669						
130	34.3384						
140	34.5424	.32	314.3	48.45	.982	.886	.32
150	34.6958						
160	34.8110						
170	34.8976	.32	382	49.03	.984	1.130	.32
180	34.9627						
200	35.0482		450	49.30	.985	1.321	.32
220	35.0964						
230	35.1119		516	49.5	.986	1.468	.32
260	35.1386						
280	35.1470						
300	35.1516						
320	35.1540						
340	35.1552	.32					
360	35.1557						
380	35.1559						
400	35.1559	.32	900	50.15**	.999	.172	.32
426	35.1559	.32					

*All $\overline{Q}_{t_D}$ values (except 50.15) correspond to area ratio of 100

**This $\overline{Q}_{t_D}$ value corresponds to the correct area ratio of 101.3

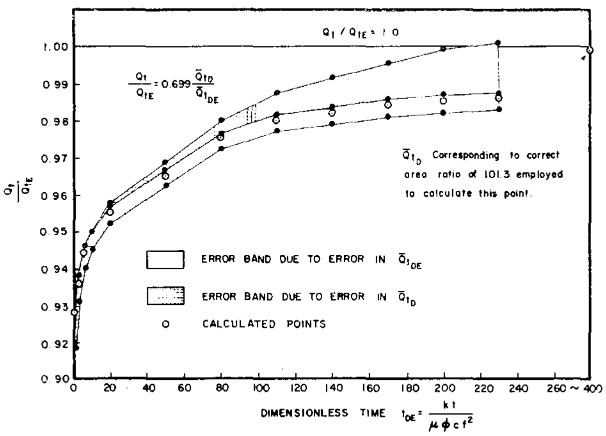

FIG. 3—COMPARISON BETWEEN ELLIPTIC AND RADIAL FLOW.

DISCUSSION OF RESULTS

From Fig. 3, the conclusion can be drawn that the error incurred by applying radial flow calculations to elliptically shaped reservoirs is inversely proportional to the magnitude of the t_D values employed in the calculations. For example, suppose an elliptically shaped reservoir (with a finite, impervious exterior boundary) is approximated by an equal area circle and Eq. 26 is employed to calculate the water influx, Q_t, for some pressure drop, Δp, at the reservoir boundary. Then the error in Q_t will be larger if the range of t_D values, for the time period of interest, is 0 to 20 than if the range is 0 to 1000. This fact is evidenced by the curve plotted in Fig. 3, since the error in the calculated Q_t value is larger for small t_D than for large t_D. The fact that the error in Q_t ranges (approximately) from 7 to 2 per cent for intermediate t_D from 2.245 ($t_{DE} = 1$) to 224.5 ($t_{DE} = 100$) indicates that a need exists for analytical solution of the elliptic flow diffusivity equation for larger (more practical) aquifer-reservoir volume ratios. Tables of dimensionless production or pressure drop quantities, analogous to the tables presented by Van Everdingen and Hurst,[2] could be developed from the analytical solution for inner elliptic boundaries of various eccentricities. Comparison of the tabulated elliptic and radial flow quantities (in a manner similar to that illustrated herein) would allow definite conclusions concerning the effect of reservoir areal shape on the field performance.

NOMENCLATURE

a = semi-major axis of ellipse, $f \cosh(u)$, feet
b = semi-minor axis of ellipse, $f \sinh(u)$, feet
c = sum of aquifer formation and fluid compressibilities, 1/psia
f = foci of ellipse are at $x = \pm f$, $y = 0$, f in feet
h = thickness or height of flow model, feet
k/μ = mobility of aquifer formation, ft²/sec-psi
p = pressure, psia
$\bar{p}$ = pressure drop, $p_o - p$, psi
p_o = initial pressure, psia
q = volumetric liquid flow rate across elliptic boundary, ft³/sec
$\bar{Q}_{tD}$ = dimensionless influx quantity for radial flow, tabulated by Van Everdingen and Hurst[2]
$\bar{Q}_{tDE}$ = dimensionless influx quantity for elliptic flow
Q_t = cumulative liquid influx into inner circle of radial flow model at time, t, ft³
Q_{tE} = cumulative liquid influx into inner ellipse of elliptic flow model at time, t, ft³
r = radius, feet
r_b = inner radius of circular flow model, feet
r_e = exterior radius of circular flow model, feet
$R = \left(\frac{\Delta u}{\Delta v}\right)^2$
s = arc length on ellipse, feet
t = time, seconds
t_D = dimensionless time for radial flow, $= \frac{kt}{\mu\phi c r_b^2}$
t_{DE} = dimensionless time for elliptic flow, $= \frac{kt}{\mu\phi c f^2}$
u,v,w = elliptic cylinder coordinates
u_b = value of u on inner elliptic boundary of flow model
u_e = value of u on exterior elliptic boundary of flow model
$\vec{V}$ = velocity vector, ft/sec
ϕ = porosity, fraction

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