Fluid Migration Across Fixed Boundaries in Reservoirs Producing by Fluid Expansion

R. E. COLLINS
MEMBER AIME

HUMBLE OIL & REFINING CO.
HOUSTON, TEX.

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024

ABSTRACT

The existence of fluid migration across fixed boundaries in oil and gas reservoirs has been known for many years. Several techniques have been developed in the past for estimating the rate of migration across fixed boundaries as an aid in planning field development and in the valuation of oilfield properties. The principal deterrent to the use of these techniques lies in the rather extensive reservoir and field data required for calculations of fluid migration. For this reason, a new, simplified procedure has been developed which makes possible the calculation of fluid migration with a minimum of field and reservoir data. This new technique is based on certain solutions of the differential equations describing flow in the reservoir which assume, for the portion of the reservoir of interest, that the formation can be approximated by a homogeneous rock of uniform thickness, that only a single mobile fluid phase exists, and that fluid production at the well is solely a result of expansion of the reservoir fluids.

The results of the present work are compiled in a set of curves. These curves can be used to calculate both rate and cumulative fluid migration when the above assumptions are justified. The only data required for such calculations are the production histories of all wells in the field, the permeability and porosity of the reservoir, the compressibility and viscosity of the fluid, and the shape of the reservoir.

INTRODUCTION

The migration of fluids across fixed boundaries in oil and gas reservoirs has long been recognized as an engineering and economic problem, but no methods for estimating the extent of migration during the development stage of a reservoir have been published.

The investigation reported here has been directed toward providing a simple technique for estimating the extent of such migration and has therefore been restricted to the simple case of a homogeneous reservoir of uniform thickness containing compressible liquids. It is assumed that only a single mobile fluid phase exists and that fluid production at the wells is solely by expansion of the reservoir fluids. Obviously, few, if any, reservoirs conform to these assumptions during their entire productive life. On the other hand, most reservoirs approximate fluid expansion reservoirs during their initial stages of primary production. Consequently, the present work should be applicable during the field development of most reservoirs, regardless of whether or not they ultimately are gas cap, dissolved-gas, or water-drive reservoirs.

The mathematical analysis of this problem has been outlined in the Appendix, while the results of the investigation are presented in the following discussion.

DISCUSSION OF CALCULATION PROCEDURE

SINGLE WELL, INFINITE RESERVOIR

The procedure for calculating fluid migrations is based on solutions of the partial differential equations describing the flow of a homogeneous compressible liquid in a homogeneous reservoir of uniform thickness. This differential equation has the form of the well-known diffusion equation. The equation, as applicable to our problem, is

$$\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} = \frac{\phi \mu c}{k} \frac{\partial \rho}{\partial t},$$

where ρ is fluid density, ϕ is porosity, μ is viscosity, c is compressibility,* and k is permeability; x, y and t are the space coordinates and time. Derivation of this equation is given in the Appendix.

The analytical solution of this equation for a single well of negligible radius producing at a constant rate, q, from a reservoir of infinite areal extent for a time, t, can be readily obtained. From this solution the rate of flow and also the cumulative flow across a straight line segment of length, l, at a perpendicular distance, δ, from the well can be computed. One end of the line must be on the perpendicular joining the well to the

*The compressibility, c, is dependent upon the volumes and compressibilities of the oil, connate water and reservoir rock. In most cases the contribution from the reservoir rock may be neglected so that c may be expressed as $c = c_o(1 - S_w) + c_w S_w$, where c_o and c_w are the oil and connate water compressibilities, respectively, and S_w is the connate water saturation.

Original manuscript received in Society of Petroleum Engineers office April 3, 1958. Revised manuscript received Nov. 19, 1958.

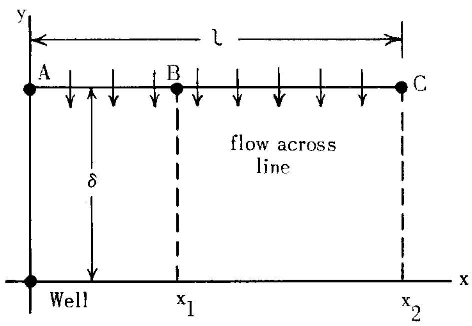


FIG. 1—DIAGRAM OF WELL OF NEGLIGIBLE RADIUS PRODUCING AT A CONSTANT RATE.

line. This is shown in Fig. 1. It is demonstrated in the Appendix that all possible values of the flow rate, $q\perp$, across the line, l, can be plotted in the form, $q\perp/q$ vs $4kt/\phi\mu c\delta^2$, with l/δ as a parameter. Here, q is the production rate of the well. Such curves are plotted in Fig. 2. Also it is found that the cumulative flow across the line, $Q\perp$, can be plotted as $Q\perp/Q$ vs $4kt/\phi\mu c\delta^2$, with l/δ as a parameter. Here, Q is the cumulative production of the well. This plot is shown in Fig. 3. Note that the units are those shown in the legend of each figure.

Using the two sets of curves in Figs. 2 and 3 and simple subtraction, the value of $q\perp$ or $Q\perp$ for a line segment such as BC in Fig. 1 can be computed. Thus,

$$\left(\frac{q\perp}{q}\right)_{BC} = \left(\frac{q\perp}{q}\right)_{AC} - \left(\frac{q\perp}{q}\right)_{AB},$$

and, similarly,

$$\left(\frac{Q\perp}{Q}\right)_{BC} = \left(\frac{Q\perp}{Q}\right)_{AC} - \left(\frac{Q\perp}{Q}\right)_{AB}.$$

Using these relations the values of $q\perp/q$ and $Q\perp/Q$ for any line segment can be obtained. Note that the values of $q\perp/q$ and $Q\perp/Q$ for the line segments AC and AB can be obtained from the curves of Figs. 2 and 3 since these line segments correspond to the case already given.

MULTIPLE WELLS, INFINITE RESERVOIR

To calculate $q\perp$ and $Q\perp$ for a given line segment in an infinite reservoir containing several wells producing at constant rates $q_1, q_2, \ldots$, where the wells are numbered 1, 2, . . j, $j + 1, \ldots$, etc., the results derived in Appendix A can be employed; namely,

$$q\perp = \sum_j q_j^* \left(\frac{q\perp}{q}\right)_j ,$$

where $(q\perp/q)_j$ is the value that $q\perp/q$ would have if only the jth well were in the reservoir and

$$q_j^* = \begin{cases} q_j, & \text{the production rate of } j\text{th well if well is below line segment, or} \\ -q_j, & \text{minus the production rate of } j\text{th well if well is above line segment.} \end{cases}$$

Note that $(q\perp/q)_j$ can be read from the curves in Figs. 2 and 3 for each well.

The value of $q\perp$ derived from the cited procedure will be positive if the net flow across the line is from above to below the line segment.

The value of $Q\perp$ for multiple wells is obtained in a similar manner, namely,

$$Q\perp = \sum_j Q_j^* \left(\frac{Q\perp}{Q}\right)_j ,$$

where Q_j^* is related to the cumulative production of the jth well like q_j^* above is related to the production

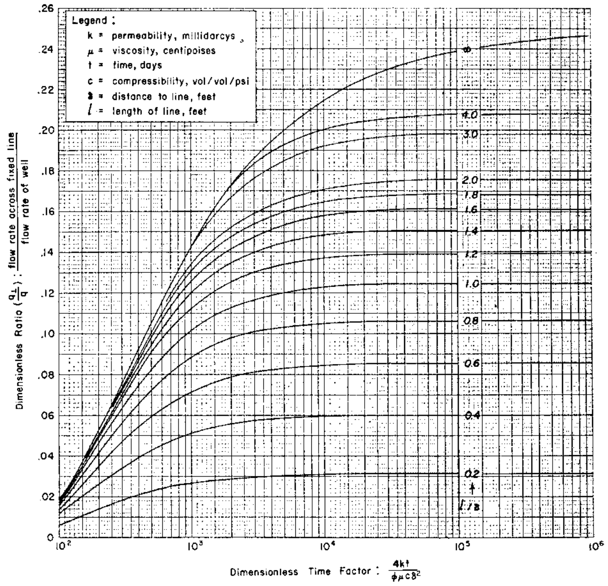


FIG. 2—FLOW RATE ACROSS FIXED LINE.

rate. Here $(Q\perp/Q)_j$ is the value of $Q\perp/Q$ which would exist if the jth well were the only well in the field.

MULTIPLE BOUNDARIES

With these tools in hand the net flow into or out of a region enclosed by straight line segments, a, b, c, etc., in an infinite reservoir containing several wells producing at constant rates can be calculated. By simple addition, $q\perp$ for the region is $q = (q\perp)_a + (q\perp)_b + (q\perp)_c + (q\perp)_d + (q\perp)_e$, and similarly for $Q\perp$. Here, flow into the region is considered positive. The flow across each line can be computed as outlined previously, taking care to use the correct sign on q_j^*. Here, q_j^* will be $+ q_j$ if the jth well causes flow into the region across the line being considered. This same procedure can be applied to compute $Q\perp$.

BOUNDED RESERVOIRS

In the case of bounded reservoirs one can employ the method of images to represent the boundaries. In practice only those wells nearest the boundary need be considered in this way. To determine whether an

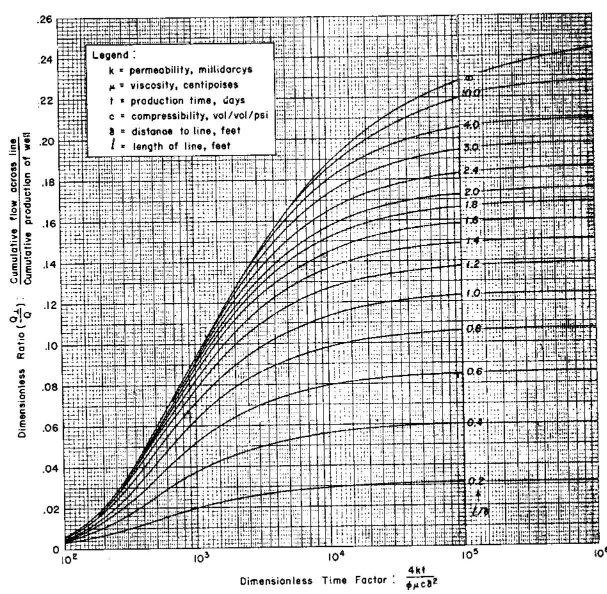
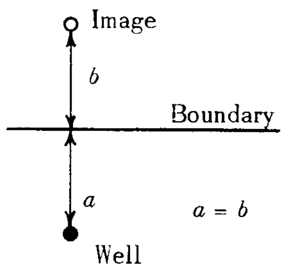


FIG. 3—CUMULATIVE FLOW ACROSS FIXED LINE.

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024

image is required for a given well one can consider the boundary as a fixed line and compute $q\perp/q$ for this line and the well being considered. If the value of $q\perp/q$ thus determined is negligibly small, then an image of this well is not required. If such is not the case, then the image well is placed as shown, with the production rate and production time of the image well being the same as those of the well. After all necessary images are located proceed as for an infinite reservoir, treating the images as real wells.



Variable Production Rates

For wells having variable rates and/or shut-in periods, the following procedure can be used.

Suppose the production rate of the well as a function of time can be approximated by step-wise changes of q. That is, the production rate during the time t_n to t_{n+1} is taken as constant and equal to the average rate for this period. Thus, the rate history can be represented by the following.

$$q = q_1 \ , \ o<t<t_1$$
$$q = q_n \ , \ t_n<t<t_{n+1} \ , \ 1<n<N$$
$$q = q_N, \ t_N<t \ .$$

The selection of the constant-rate periods should be made so that the actual rate is nearly constant in the period. That is, the t_n's are selected as those times at which the rate changes sharply. In the Appendix, section titled, "Variable Rates; Single Fixed Line," it is shown that for this case $q\perp$ is given by

$$q\perp = \sum_{n=1}^{N} (q_n - q_{n-1}) \left(\frac{q\perp}{q}\right)_n \ ,$$

where $(q\perp/q)_n$ is the value $q\perp/q$ would have if the rate were constant and the production time were $t - t_{n-1}$, $t>t_N$.

The same analysis applies to $Q\perp$, namely,

$$Q\perp = \sum_{n=1}^{N} \left(\frac{Q\perp}{Q}\right)_n \bar{Q}_n \ ,$$

where Q_n is given by $\bar{Q}_n = (q_n - q_{n-1}) \ \ (t - t_{n-1})$. Here, $(Q\perp/Q)_n$ is the value $Q\perp/Q$ would have for constant rate and production time, $t - t_{n-1}$.

With the procedures given here, a wide variety of migration problems can be solved. Two examples of such problems are treated in the next section.

EXAMPLES

Example Problem No. 1

As the first example illustrating the techniques involved in calculating fluid migration across fixed lines, the well geometry shown in Fig. 4 is considered. Note that since Well No. 2 is very close to the reservoir boundary an image well is included, i.e., No. 2′.

The pertinent data for this problem are: k = 25 md; $\phi = 0.25$; $c_o = 5.01 \times 10^{-4}$ vol/vol/psi; $c_w = 3 \times 10^{-6}$ vol/vol/psi; $S_w = 0.20$; $\mu = 2$ cp; and distance OA = 1,600 ft.

TABLE 1—PERTINENT DATA FOR EXAMPLE PROBLEM NO. 1

Well No.	X (ft)	Y (ft)	q (B/D)	Time of completion (days)	Production time (days)
1	1,400	50	150	0	t
2	200	200	100	60	t−60
3	1,400	−50	100	160	t−160
4	600	−300	100	220	t−220
2′	−200	200	100	60	t−60

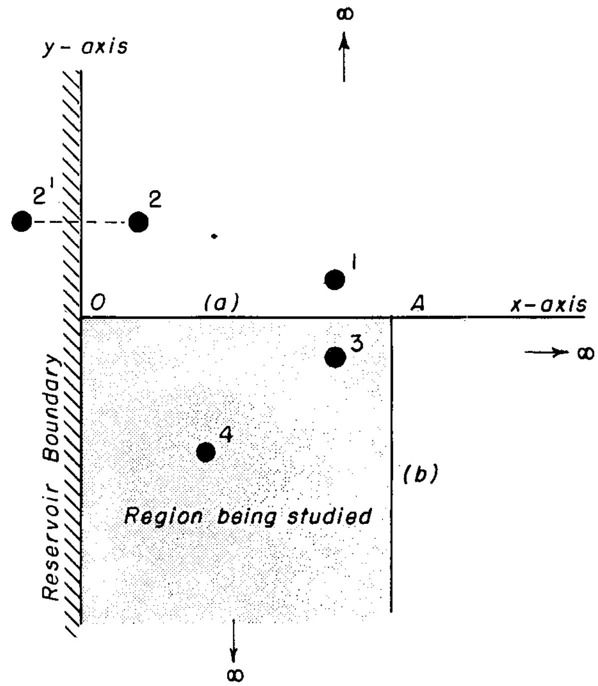


FIG. 4—WELL GEOMETRY FOR EXAMPLE NO. 1.

In this analysis flow rates and cumulative production occur as ratios; hence, if the q's and Q's are expressed at reservoir conditions, the $q\perp$'s and $Q\perp$'s will be at these conditions also.

The effective compressibility is computed as, $c = c_o (1 - S_w) + c_w S_w = .8(5.01 \times 10^{-4}) + .2(3 \times 10^{-6}) = 4 \times 10^{-4}$ vol/vol/psi.

Now, to calculate the net flow into the indicated region at several different times: first, t = 160 days, when the first well in this region is completed, second, at t = 220 days when the second well in the region is completed, and finally at t = 320 days or 100 days after the second well in the region is completed.

To illustrate the method in detail, the flow across line (a) due to Well No. 1 at time t = 160 days is calculated. First, a perpendicular line is constructed from this well to line (a). This gives a line segment to the right of the perpendicular of length 200 ft and a line to the left of length 1,400 ft. For the line to the right, l/δ = 200/50 = 24, and using the units specified in Fig. 2,

$$\frac{4kt}{\phi\mu c\delta^2} = \frac{4 \times 25 \times 160}{0.25 \times 2 \times 4 \times 10^{-4} \times (50)^2} = 3.2 \times 10^4 \ .$$

Using the curves of $Q\perp/Q$ in Fig. 3, it is found that $Q\perp/Q = 0.197$. For the left line segment, l/δ = 1,400/50 = 28, and the same value for the time group, $4kt/\phi\mu c\delta^2$. From the curve, Fig. 3, it is found that $Q\perp/Q$ = 0.212. Thus, the total flow across line (a) due to Well No. 1 is $(Q\perp/Q)_{1a} = 0.197 + 0.212 = 0.409$.

The cumulative production of Well No. 1 at this time is $Q = q_1 t = 150 \times 160 = 24{,}000$ bbl, and since this well causes flow *out* of the region being considered, $Q^*_1 = -Q_1 = -24{,}000$ bbl. Then,

$$(Q\perp)_{1a} = Q^*_1 \left(\frac{Q\perp}{Q}\right)_{1a} = -24{,}000 \times 0.409 \ ,$$

or

$$(Q\perp)_{1a} = -9{,}816 \text{ bbl.}$$

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024

TABLE 2—CALCULATIONS

Line	Well No.	Part line (l/δ)	Production time (days)	$\frac{4kt}{\phi \mu c \delta^2}$	$Q\perp/Q$ Part line	$Q\perp/Q$ Whole line	Q_j,* (bbl)	$Q\perp ij$ (bbl)
(a)	1	28	160	3.2×10^4	0.212			
(a)	1	4	160	3.2×10^4	0.197	0.409	−24,000	−9,816
(a)	2	1	100	1.25×10^3	0.078			
(a)	2	7	100	1.25×10^3	0.104	0.182	−10,000	−1,820
(a)	2	9	100	1.25×10^3	0.104			
(a)	2′	1	100	1.25×10^3	−0.078	0.026	−10,000	− 260
(b)	1	0.25	160	2×10^3	−0.030			
(b)	1	∞	160	2×10^3	0.126	0.096	24,000	+ 960
(b)	2	0.1429	100	2×10^1	0			
(b)	2	∞	100	2×10^1	0	0	10,000	0
(b)	2′	0.1111	100	2×10^1	0			
(b)	2′	∞	100	2×10^1	0	0	10,000	0

Net $Q\perp$ = − 10,936 bbl

To compute $(Q\perp)_{1b}$, the flow across line (b) due to Well No. 1, a perpendicular is constructed from the well to the extension of line (b) and proceed as for line (a).

Here it is necessary to calculate the flow across the semi-infinite line starting at A and going downward and subtract from it the flow across the extended portion of line (b). Note that here $(Q\perp)_{1b}$ will turn out positive because Well No. 1 causes flow *into* the region. The calculations which were made 160 days after completion of Well No. 1 are summarized in Table 2. Proceeding in the manner outlined here, the desired $Q\perp$ is obtained 220 and 320 days after completion of Well No. 1. The results of these calculations are summarized in Table 3.

TABLE 3—SUMMARY OF CALCULATIONS

Production time of well No. 1 days)	Net flow into region (bbl)
160	−10,936
220	−11,358
320	−11,942

Thus, the region is continually losing fluid to adjacent areas.

Example Problem No. 2

The second example considered here is not presented in detail, only the final results are given to illustrate the time dependence of fluid migration across fixed lines. In this case a single fixed line of essentially infinite length is considered with the well geometry shown in Fig. 5. The pertinent data are $k = 25$ md; $\phi = 0.25$; $\mu = 0.20$; and $c = 4 \times 10^{-4}$ vol/vol/psi (effective value).

TABLE 4—PERTINENT DATA FOR EXAMPLE PROBLEM NO. 2

Well No.	q (B/D)	Completion time (days)	Production time (days)	δ (ft)
1	150	0	t	200
2	100	100	$t - 100$	100
3	100	200	$t - 200$	50

Here, positive flow is toward Well No. 1.

Results of the calculations for this problem are shown in Fig. 6 where plots of $q\perp$ and $Q\perp$ vs time are given. Note that when Well No. 2 is put on production $q\perp$ starts to decrease rapidly. It appears that $q\perp$ would have stabilized at a positive value, and this is actually true, if Well No. 3 had not been put on production. However, $q\perp$ decreases again when Well No. 3 is put on production and finally stabilizes at a negative value. The effect of these variations in $q\perp$ on $Q\perp$ is shown in Fig. 6.

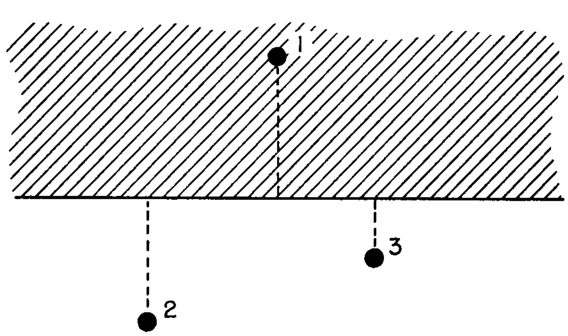


Fig. 5—Well Geometry for Example No. 2.

CONCLUSIONS

The method presented in this paper for calculating fluid migration across fixed lines is simple, rapid, and requires a minimum of reservoir data. Proper application of such calculations should be of great benefit in valuation of properties and planning field development.

APPENDIX

MATHEMATICAL THEORY

Single Well; Constant Rate; Single Fixed Line

The equation of continuity, which assures the conservation of mass, is

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) = -\phi \frac{\partial \rho}{\partial t} \quad . \quad . \quad . \quad (1)$$

where ρ is the fluid density, ϕ is the porosity of the medium, t is the time and v_x and v_y are the x and y components of velocity, respectively. It is assumed here that the porous medium is uniform, the flow is two dimensional and the fluid is homogeneous. Combining this equation with Darcy's law yields

$$\frac{\partial}{\partial x}\left(\rho \frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\rho \frac{\partial p}{\partial y}\right) = \frac{\phi\mu}{k}\frac{\partial \rho}{\partial t} \quad . \quad . \quad (2)$$

where k is the permeability of the medium and μ is the viscosity of the fluid. Then, if the fluid density is related to the pressure by

$$\rho = \rho_i \, e^{c(p-p_i)}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where ρ is fluid density at pressure p, ρ_i is density at pressure p_i, and c is the compressibility, Eq. 2 can be written as

$$\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} = \frac{\phi\mu c}{k}\frac{\partial \rho}{\partial t} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The relationship given in Eq. 3 is the correct pressure dependence for normal compressible liquids.

The solution of Eq. 4 corresponding to a well of

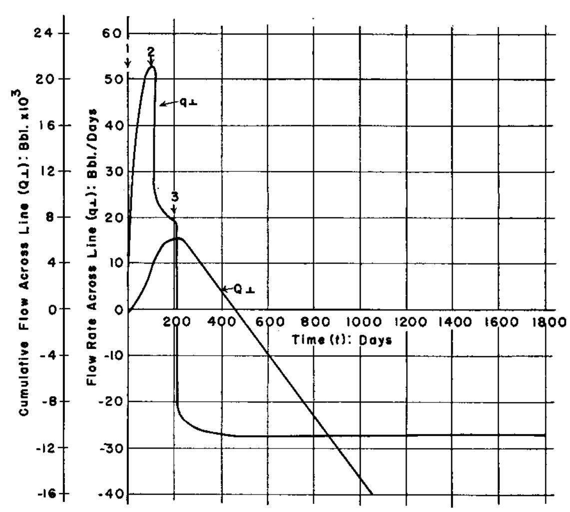


Fig. 6—Flow Across a Fixed Line (Example No. 2).

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024

negligible radius, at $x = 0$, $y = 0$, producing at the constant rate q (measured at initial reservoir pressure and temperature) for a time t, in a reservoir of essentially infinite areal extent, is[2]

$$\frac{\rho(x,y,t)}{\rho_i} = 1 - \frac{q\mu c}{4\pi kh}\int_{\frac{\phi\mu c(x^2+y^2)}{4kt}}^{\infty}\frac{e^{-\xi}}{\xi}\,d\xi \quad . \quad . \quad . \quad . \quad (5)$$

Note in this solution the p_i of Eq. 3 is taken as the initial reservoir pressure and ρ_i is the fluid density at that pressure. The units used here and throughout this section are consistent units unless stated explicitly otherwise.

With this solution of the flow equation as a basic tool proceed to consider the migration of fluid in the reservoir across a straight-line segment of length l at a *perpendicular* distance, δ, from a well producing at constant rate, q. This line segment, AB, is shown in relation to the well in Fig. 1. The mass rate of flow, m, (toward the well) across the line (gm/sec) is given by

$$m = \int_o^l \frac{kh}{\mu}\rho(x,\delta,t)\frac{\partial p(x,\delta,t)}{\partial y}\,dx \quad . \quad . \quad . \quad (6)$$

or, the volume rate (at inital reservoir pressure) is

$$q\perp = \frac{m}{\rho_i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

Eq. 6 is simply a direct application of Darcy's law.

Employing Eq. 5 for $\rho(x, y, t)$ to obtain

$$\frac{\partial\rho}{\partial y} = c\rho\frac{\partial p}{\partial y} = \frac{q\mu c\rho_i}{2\pi kh}\,y\,\frac{e^{-a(x^2+y^2)/t}}{x^2+y^2}, \quad . \quad (8)$$

where for convenience the notation,

$$a = \frac{\phi\mu c}{4k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

is introduced. Then, substituting Eq. 8 into Eq. 7 with m given by Eq. 6, yields

$$q\perp = \frac{q}{2\pi}\int_o^l \frac{\delta e^{-a(x^2+\delta^2)/t}}{x^2+\delta^2}\,dx, \quad . \quad . \quad . \quad (10)$$

which is an analytical expression for the rate of fluid migration across the line segment that is subject to evaluation.

Also an expression for the cumulative flow across this line segment can be written simply by integrating Eq. 10 with respect to the time, t, thus,

$$Q\perp = \frac{Q}{2\pi t}\int_o^t\int_o^l \frac{\delta e^{-a(x^2+\delta^2)/\theta}\,dx}{x^2+\delta^2}\,dx\,d\theta \quad . \quad . \quad (11)$$

Before proceeding to the evaluation of these integrals they can be written in a more useful form; thus, by changing variables in the integrals,

$$\frac{q\perp}{q} = \frac{1}{2\pi}\int_o^\beta \frac{e^{-1(\xi^2+1)/\alpha}}{\xi^2+1}\,d\xi, \quad . \quad . \quad . \quad . \quad (12)$$

and

$$\frac{Q\perp}{Q} = \frac{1}{2\pi\alpha}\int_o^\alpha\int_o^\beta \frac{e^{-1(\xi^2+1)/\mu}}{\xi^2+1}\,d\xi\,d\mu, \quad . \quad . \quad (13)$$

where

$$\beta = \frac{l}{\delta} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

and

$$\alpha = \frac{4kt}{\phi\mu c\delta^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Here, note that the units of q, $q\perp$, Q and $Q\perp$ are immaterial; that is, $q\perp/q$ is dimensionless as is $Q\perp/Q$, thus the flow rate, $q\perp$, can be expressed in, say, surface barrels per day simply by expressing q in these units. The same holds true for $Q\perp$ and Q.

For a line of infinite length, $l = \beta = \infty$, the cited integrals can be evaluated analytically. Thus,

$$\left(\frac{q\perp}{q}\right)_{l=\infty} = \frac{1}{4}\,\mathrm{erfc}\sqrt{\frac{\phi\mu c\delta^2}{4kt}} \quad . \quad . \quad . \quad (16)$$

and

$$\left(\frac{Q\perp}{Q}\right)_{\beta=\infty} = \frac{1}{4}\left\{\left(1+\frac{2}{\alpha}\right)\mathrm{erfc}\frac{1}{\sqrt{\alpha}} - \frac{2}{\sqrt{\pi\alpha}}e^{-1/\alpha}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

where, as always, α is given by Eq. 15 and erfc denotes the error function compliment defined as

$$\mathrm{erfc}\,u = \frac{1}{\sqrt{\pi}}\int_u^\infty e^{-\lambda^2}\,d\lambda \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

For finite β the integrals in Eqs. 12 and 13 are not very tractable, hence a numerical integration must be employed. The integral in Eq. 12,

$$\int_o^\beta \frac{e^{-1/\alpha(\xi^2+1)}}{\xi^2+1}\,d\xi,$$

is in a form satisfactory for numerical integration; this has been carried out on a medium speed magnetic drum computer for a wide range of values of α and many values of β.

Before applying numerical techniques to Eq. 13 it must, for practical reasons, be reduced from a double integral to one or more single integrals; this is accomplished by suitable changes of variable and also interchanging the order of integration.

Thus, obtain

$$\frac{Q\perp}{Q} = \frac{\beta}{\pi}\int_o^\infty \frac{ze^{-(z^2+1)/\alpha}}{(z^2+1)^2}\,dz - \frac{\beta}{\pi}\int_o^\beta \frac{ze^{-(z^2+1)/\alpha}}{(z^2+1)^2}\,dz + \frac{1}{\pi}\int_o^\beta \frac{z^2e^{-(z^2+1)/\alpha}}{(z^2+1)^2}\,dz. \quad . \quad . \quad . \quad . \quad . \quad (19)$$

The first integral here can be evaluated analytically. Thus,

$$\frac{\beta}{\pi}\int_o^\infty \frac{ze^{-(z^3+1)/\alpha}}{(z^2+1)^2}\,dz = \frac{\beta}{\pi}e^{-1/\alpha} - \frac{\beta}{\pi\alpha}\left\{-Ei\left(-\frac{1}{\alpha}\right)\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

where the Ei — function is defined as

$$-Ei\left(-\frac{1}{\alpha}\right) = \int_{1/\alpha}^\infty \frac{e^{-\mu}}{\mu}\,d\mu \quad . \quad . \quad . \quad . \quad (21)$$

Eq. 19 with the first integral given by Eq. 20 is in suitable form for numerical evaluation. The last two integrals in Eq. 19 have been evaluated numerically on the computer for a wide range of values of α and β.

As mentioned previously, the units for q, $q\perp$, Q and $Q\perp$ in the equations involving ratios of these quantities are arbitrary. Similarly, since $\beta = l/\delta$ is dimensionless,

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024

the units of l and δ are arbitrary; it is only required that l and δ have the same units. To express α in "oilfield units" (as given on the curves in Figs. 2 and 3), have

$$\left(\frac{4kt}{\phi\mu c\delta^2}\right)_{\text{oilfield}} = 0.158\left(\frac{4kt}{\phi\mu c\delta^2}\right)_{\text{cgs}} \quad . \quad (22)$$

The calculated values of $q\perp/q$ and $Q\perp/Q$, as obtained by the combination of analytical and numerical techniques, are plotted vs α with β as parameter in Figs. 2 and 3, respectively.

As t and, hence, α approach infinity, $Q\perp/Q$, according to Eq. 19, approaches

$$\left(\frac{Q\perp}{Q}\right)_{\alpha=\infty} \frac{\beta}{\pi}\int_{\beta}^{\infty}\frac{zdz}{(z^2+1)^2} + \frac{1}{\pi}\int_{o}^{\beta}\frac{z^2dz}{(z^2+1)^2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (23)$$

These integrals can be evaluated to give,

$$\left(\frac{Q\perp}{Q}\right)_{\alpha=\infty} = \frac{1}{2\pi}\tan^{-1}\beta \quad . \ . \ . \ . \ . \quad (24)$$

Thus, if $Q\perp/Q$ is plotted vs $(\tan^{-1}\beta)/2\pi$ with α as parameter, the curves for large α should approximate straight lines. This result simplifies interpolation. Similarly, one can also show that

$$\left(\frac{q\perp}{q}\right)_{\alpha=\infty} = \left(\frac{Q\perp}{Q}\right)_{\alpha=\infty} \quad . \ . \ . \ . \quad (25)$$

If it is desired to determine $q\perp$ and/or $Q\perp$ across a line BC such as shown in Fig. 1, note that by simple addition of integrals in Eqs. 12 and 13,

$$\frac{q\perp}{q}(x_2 - x_1, \delta, \alpha) = \frac{q\perp}{q}(\beta_2, \delta, \alpha) - \frac{q\perp}{q}(\beta_1, \delta, \alpha)\ , \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (26)$$

and similarly for $Q\perp/Q$. That is the value of $q\perp/q$ corresponding to a line from $x = 0$ to $x = l_2 = x_2$ is read from the curves given in Fig. 2 and from this is subtracted the value for $q\perp/q$ corresponding to a line from $x = 0$ to $x = l_1 = x_1$. The remainder is the flow rate across the line from $x = x_1$ to $x = x_2$. The same procedure applies to $Q\perp/Q$.

Multiple Wells; Constant Rates; Single Fixed Line

By a well-known superposition theorem[2] the solution of the flow equation (Eq. 4) for an infinite reservoir containing N point sinks (wells of negligible radii) with production rate q_n, $n = 1, 2 \ldots N$, and production times t_n, $n = 1, 2, 3, \ldots N$, is

$$\frac{\rho(x,y,t)}{\rho_i} = 1 - \frac{\mu c}{4\pi kh}\sum_{n=1}^{N} q_n \int_{\frac{\phi\mu c r_n^2}{4kt_n}}^{\infty}\frac{e^{-\xi}}{\xi}d\xi\ , \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad (27)$$

where

$$r_n^2 = (x - x_n)^2 + (y - y_n)^2 \quad . \ . \ . \ . \quad (28)$$

x_n and y_n being the x and y coordinates of the nth well.

It is desired to calculate the rate of migration across the line from $x = 0$ to $x = l$ at $y = \delta$ for this system of wells. Positive flow is assumed in the direction of decreasing y. Eq. 6,

$$m = \int_{o}^{l}\frac{kh}{\mu}\rho(x,\delta,t)\frac{\partial p}{\partial y}(x,\delta,t)\,dx,$$

is still valid. Substituting Eq. 27 into Eq. 6 gives, with Eq. 7,

$$q\perp = \frac{1}{2\pi}\sum_{n=1}^{n} q_n(\delta - y_n) \int_{o}^{l}\frac{e^{-\alpha/t_n[(x-x_n)^2+(\delta-y_n)^2]}}{[(x-x_n)^2+(\delta-y_n)^2]}dx\ , \quad . \ . \ . \quad (29)$$

for the rate of migration across the line. A new symbol, q_n^*, is introduced as follows.

$$q_n^* = q_n \text{ if } \delta > y_n$$
$$q_n^* = -q_n \text{ if } \delta < y_n \quad . \ . \ . \ . \ . \ . \ . \quad (30)$$

Then have,

$$q\perp = \frac{1}{2\pi}\sum_{n=1}^{N} q_n^*\,|\delta - y_n| \int_{o}^{l}\frac{e^{-\alpha/t_n[(x-x_n)^2+(\delta-y_n)^2]}}{(x-x_n)^2+(\delta-y_n)^2}dx\ , \quad . \ . \ . \quad (31)$$

and noting that $|\delta - y_n|$ is the distance of the nth well from the line, always taken positive, define

$$\delta_n = |\delta - y_n|\ , \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (32)$$

and then have

$$q\perp = \frac{1}{2\pi}\sum_{n=1}^{N} q_n^* \int_{-x_n/\delta_n}^{l/\delta_n}\frac{e^{-1/\alpha_n(\xi^2+1)}}{\xi^2+1}d\xi \quad . \ . \quad (33)$$

Thus, see that $q\perp$ is just the algebraic sum of the flows caused by each point sink, i.e., to calculate the rate of migration across the line consider each well as though it were the only well in the field and compute $q\perp q$. Then, take these $(q\perp/q)_n$, $N = 1, 2 \ldots N$, and compute.

$$(q\perp)_n = \left(\frac{q\perp}{q}\right)_n \cdot q_n^*\ , \quad . \ . \ . \ . \ . \ . \quad (34)$$

and then form the sum,

$$q\perp = \sum_{n=1}^{N} q_n^*\left(\frac{q\perp}{q}\right)_n \quad . \ . \ . \ . \ . \ . \ . \quad (35)$$

for the rate of migration across the line due to N wells.

In a similar manner it can be shown that $Q\perp$ for N wells is given by

$$Q\perp = \sum_{n=1}^{N} Q_n^*\left(\frac{Q\perp}{Q}\right)_n \quad . \ . \ . \ . \ . \ . \quad (36)$$

where

$$\begin{cases} Q_n^* = Q_n & y_n < \delta \\ Q_n^* = -Q_n & y_n > \delta \end{cases} \quad . \ . \ . \ . \ . \ . \ . \quad (37)$$

and $(Q\perp/Q)_n$ is the value of $Q\perp/Q$ which would result if the nth well were the only well in the field.

Variable Rates; Single Fixed Line

The multiple well solution, Eq. 27, for constant rates, q_n, $n = 1, 2 \ldots N$, can be used to construct the solution for a single well whose production rate is varied in a step-wise manner. Suppose the production history of the well to be

$$\begin{cases} q = q_1 \text{ for } 0 < t < t\ , \\ q = q_n \text{ for } t_n < t < t_{n+1}\ ,\ n = 2, 3 \ldots N-1 \\ q = q_N \text{ for } t_N < t \end{cases} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad (38)$$

Then all that is needed is to consider the single well

[2]References given at end of paper.

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024

with variable rate given by Eq. 38 as N different wells with rates and production times given by

$$\begin{cases} \text{No. } 1: q_1, t \\ \text{No. } n: q_n - q_{n-1}, t - t_{n-1}, n = 2, 3 \ldots N, \end{cases} \qquad (39)$$

all located at the same point in the reservoir. Following this procedure the solution of the flow equation is obtained for a single well with rate history given by Eq. 38 as

$$\frac{p(x,y,t)}{p_i} = 1 - \frac{\mu c}{4\pi kh} \sum_{1}^{N} (q_n - q_{n-1})$$

$$\int_{\frac{\phi\mu c r^2}{4k(t - t_{n-1})}}^{\infty} \frac{e^{-\lambda}}{\lambda} d\lambda \ , \quad \ldots \ldots \ldots \ldots \quad (40)$$

where

$$r^2 = x^2 + y^2 \ , \quad \ldots \ldots \ldots \ldots \quad (41)$$

i.e., the well is assumed at the origin.

Thus, proceed as in the multiple well analysis to show that for the variable rate well,

$$q\perp = \sum_{n=1}^{N} \left(\frac{q\perp}{q}\right)_n (q_n - q_{n-1}) \ , \quad \ldots \quad (42)$$

where now $(q\perp/q)_n$ is the value of $q\perp/q$ for a single well with constant rate with $\beta = l/\delta$ and the production time, $t - t_{n-1}$, $t > t_N$.

For multiple wells, each with variable rate, each well is treated in the manner outlined here to yield a value of $(q\perp)_j$ for the jth well, then define

$$\begin{cases} (q^*\perp)_j = (q\perp)_j \ , \text{ if } y_j < \delta \\ (q^*\perp)_j = -(q\perp)_j \text{ if } y_j > \delta \end{cases} \quad \ldots \quad (43)$$

Finally, then, for multiple wells with variable rates,

$$q\perp = \sum_{j} (q^*\perp)_j . \quad \ldots \ldots \ldots \quad (44)$$

This procedure can be extended to determine $Q\perp$ for wells having variable rates as was done for multiple wells in the previous section of this Appendix. The result is

$$Q\perp = \sum_{n=1}^{N} \left(\frac{Q\perp}{Q}\right)_n \overline{Q}_n \quad \ldots \ldots \quad (45)$$

where $(Q\perp/Q)_n$ is the value $Q\perp/Q$ would have for a well with constant rate and production time, $t - t_{n-1}$. Here, $\overline{Q}_n$ is given by

$$\overline{Q}_n = (q_n - q_{n-1}) \ (t - t_{n-1}) \quad \ldots \ldots \quad (46)$$

It should be noted that any shut-in periods are also handled in this manner simply by making q_n zero for the shut-in period.

REFERENCES

1. Hall, Howard N.: "Compressibility of Reservoir Rocks," *Trans.* AIME (1953) **198,** 309.
2. Carslaw, H. S. and Jaeger, J. C.: *Conduction of Heat in Solids,* Oxford U. Press, 1947.

★★★

Downloaded from http://onepetro.org/TRANS/article-pdf/216/01/78/2176615/spe-1082-g.pdf by guest on 25 April 2024